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# Hamilton-Jacobi Formulation of Green-Schwarz Superstring

**H.A. Elegla · N.I. Farahat**

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**Abstract** The canonical formalism for fermion string 'superstring' theory is constructed by following the Hamilton-Jacobi prescription. The Hamiltonian mechanics of the Green-Schwarz superstring are discussed. The complete set of constraints is presented, implies appearance of mixed first and second class fermionic constraints, which are treated with no need to introduce gauge fixing terms. The equations of motion are derived as total differential equations in many variables. Besides, a mechanical model of $D = 11$ superstring is presented.

**Keywords** Constrained systems · Hamilton-Jacobi formalism · Supersymmetric · Superstring

## 1 Introduction

The Hamiltonian mechanics of fermionic systems is a useful tool in quantization methods of such systems. The fermionic system contains mixed first and second class constraints, which are difficult to separate by applying Dirac's formalism of constrained systems [1]. The schemes of quantization of dynamical systems with second class constraints are based on conversion of original constraints into first class ones. In Dirac bracket approach, the conversion is achieved by modification of Poisson brackets [2–5]. Another way to approach the problem is given by Batalain and Fradkin: the BFV-BRST operator quantization method [6–9]. In the BRST method one introduces auxiliary variables and constructs the BRST charge which is first class. The auxiliary variables are also used in the conversion scheme [10]. The alternative method is Hamilton-Jacobi formalism which developed by Güler [11, 12], to write down the Hamilton-Jacobi equations for singular systems and make use of its

H.A. Elegla · N.I. Farahat (✉)
Physics Department, Islamic University of Gaza, P.O. Box 108, Gaza, Palestine
e-mail: nfarahat@iugaza.edu.ps

H.A. Elegla
e-mail: helegla@iugaza.edu.ps

singularity to obtain the equations of motion as total differential equations in many variables. Our aim is to study the dynamics of supersymmetric string by the aid of Hamilton-Jacobi treatment. This method has proved their efficiency in solving the constraints that contain all kinds of restrictions [13–23]. Recently, Hamilton-Jacobi formalism is applied to study the dynamics for the superparticle with spacetime supersymmetry with different models which contain a mixed constraints of first-class and second constraints [24–26]. The Green-Schwarz superstring is the most important and interesting model of superstring theory, but this model is very difficult in studying the quantization techniques. One quantization approach is based on gauge-fixing the fermionic symmetries to get the "semi-light-cone" gauge, using the generalized Hamiltonian formalism of Batalin. Fradkin and Vilkovisky, but this action cannot be easily quantized since the gauge-fixing is difficult in appointment [27–30]. Another approach to quantize the covariant Green-Schwarz action is based on replacing the fermionic second- class constraints with an appropriate set of first-class constraints [31]. In this paper, a new formalism for the superstring will be presented, which is considered as one of the easiest methods of Hamiltonian treatment to obtain equations of motion. The advantage of the Hamilton-Jacobi formalism is that we have no need to distinguish between first and second class constraints and we do not need gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations.

The work is organized as follows: In Sect. 2, Hamilton-Jacobi Formulation is presented. In Sect. 3, Hamilton-Jacobi Formulation is proposed to analyze the classical dynamics of Green-Schwarz superstring action. The dynamics of mechanical $D = 11$ superstring model is analyzed by using Hamilton-Jacobi formulation in Sect. 4. In Sect. 5 the conclusion is given.

## 2 Hamilton-Jacobi Formalism

The system that is described by the Lagrangian $L(q_i, \dot{q}_i, t)$ or $(L(\phi, \partial\phi)$ in field theory), $i = 1, \ldots, n$, is constrained system if the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \ldots, n, \tag{1}$$

has a rank $(n - p)$, $p < n$. In this case we have $p$ momenta are dependent of each other. The generalized momenta $P_i$ corresponding to the generalized coordinates $q_i$ are defined as,

$$P_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, \ldots, n - p, \tag{2}$$

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - p + 1, \ldots, n. \tag{3}$$

Since the rank of the Hess matrix is $(n - p)$, one may solve (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, P_b) \equiv \omega_a. \tag{4}$$

Substituting (4) into (3), we obtain relations in $q_i$, $P_a$, $\dot{q}_\nu$ and $t$ in the form

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}_{|_{\dot{q}_a = \omega_a}} = -H_\mu(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, P_a, t). \tag{5}$$

The canonical Hamiltonian $H_0$ is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, t) + P_a\omega_a + \dot{q}_\mu P_{\mu}{}_{|_{P_\nu=-H_\nu}}. \tag{6}$$

The set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

$$H'{}_\alpha\left(q_\beta; q_a; P_a = \frac{\partial S}{\partial q_a}; P_\mu = \frac{\partial S}{\partial q_\mu}\right) = 0, \quad \alpha, \beta = 0, 1, \dots, p, \tag{7}$$

where

$$H'_0 = P_0 + H_0; \tag{8}$$

$$H'_\mu = P_\mu + H_\mu \tag{9}$$

with $q_0 = t$ and $S$ being the action. The equations of motion are obtained as total differential equations in many variables such as,

$$dq_a = \frac{\partial H'_\alpha}{\partial P_a} dt_\alpha, \tag{10}$$

$$dP_r = \frac{\partial H'_\alpha}{\partial q_r} dt_\alpha, \quad r = 0, 1, \dots, n. \tag{11}$$

$$dZ = \left(-H_\alpha + P_a \frac{\partial H'_\alpha}{\partial P_a}\right) dt_\alpha, \tag{12}$$

where $Z = S(t_\alpha, q_a)$. These equations are integrable if and only if [11]

$$dH'_0 = 0, \tag{13}$$

$$dH'_\mu = 0, \quad \mu = n - p + 1, \dots, n. \tag{14}$$

If the conditions (13) and (14) are not satisfied identically, we consider them as new constraints and we examine their variations. Thus repeating this procedure, one may obtain a set of constraints such that all the variations vanish, taking into account if the system is completely (where the set of equations of motion and the action function is integrable) or partially (where the set of equations of motion is only integrable) integrable system [32, 33].

## 3 Hamilton-Jacobi Formulation of Green-Schwarz Superstring

A form of the action describing the motion of a superstring in *11*-dimensional Minkowski space-time has been proposed in [34] as,

$$\begin{aligned} S = \int d^2\sigma \Bigg\{ & \frac{-g^{ab}}{2\sqrt{-g}} \Pi^\mu_a \Pi_{b\mu} - i\varepsilon^{ab}\left(\partial_a x^\mu - \frac{i}{2}\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_a \theta\right)\left(\bar{\theta}\Gamma^\mu \partial_b \theta\right) \\ & - \varepsilon^{ab}\xi_a\left(n_\mu \Pi^\mu_b\right) - n_\mu \varepsilon^{ab}\partial_a A^\mu_b - \phi(n^2+1)\Bigg\}, \end{aligned} \tag{15}$$

where $\Pi^\mu_a \equiv \partial_a x^\mu - i(\bar{\theta}\Gamma^{\nu\mu} n_\nu \partial_a \theta)$.

The quantities which appear in the Green-Schwarz action are the two-dimensional metric $g^{ab}$, a ten dimensional position $x^\mu$, $\theta$ is a *32*-component anticommuting Majorana spinor of $SO(2,9)$; which is a rotation group, $n^\mu(\sigma,\tau)$ is $D11$ vector, $\xi_a$ is a $d=2$ vector, $A_a^\mu(\sigma,\tau)$ is $D11$ vector, and $\phi(\sigma,\tau)$ is a scalar. As for the $\Gamma$-matrices, we employ the 32-dimensional Majorana representation and denote them by $\Gamma^{\mu\nu}$. In (15) we have set $\varepsilon^{ab}=-\varepsilon^{ba}$ , $\varepsilon^{01}=-1$.

The Lagrangian density is

$$\begin{aligned}\mathcal{L}=\bigg\{&\frac{-g^{ab}}{2\sqrt{-g}}\Pi_a^\mu\Pi_{b\mu}-i\varepsilon^{ab}\bigg(\partial_a x^\mu-\frac{i}{2}\bar\theta\Gamma^{\mu\nu}n_\nu\partial_a\theta\bigg)\big(\bar\theta\Gamma^\mu\partial_b\theta\big)\\&-\varepsilon^{ab}\xi_a\big(n_\mu\Pi_b^\mu\big)-n_\mu\varepsilon^{ab}\partial_a A_b^\mu-\phi(n^2+1)\bigg\}.\end{aligned}\tag{16}$$

In more explicit form, the Lagrangian density (16) takes the form

$$\begin{aligned}\mathcal{L}=-\frac{1}{2\sqrt{-g}}\bigg\{&g^{00}\Big((\partial_0x^\mu\partial_0x_\mu)-2i\big(\bar\theta\Gamma^{\nu\mu}n_\nu\partial_0\theta\big)(\partial_0x_\mu)\Big)\\&+2g^{01}\big(\partial_0x^\mu\partial_1x_\mu-i(\bar\theta\Gamma^{\nu\mu}n_\nu\partial_0\theta)(\partial_1x_\mu)-i(\bar\theta\Gamma^{\nu\mu}n_\nu\partial_1\theta)(\partial_0x_\mu)\big)\\&+g^{11}\big((\partial_1x^\mu\partial_1x_\mu)-2i(\bar\theta\Gamma^{\nu\mu}n_\nu\partial_1\theta)(\partial_1x_\mu)\big)\bigg\}+i(\bar\theta\Gamma_\mu\partial_1\theta)(\partial_0x^\mu)\\&-i(\bar\theta\Gamma_\mu\partial_0\theta)(\partial_1x^\mu)+\xi_0n_\mu\big((\partial_1x^\mu)-i(\bar\theta\Gamma^{\nu\mu}n_\nu\partial_1\theta)\big)\\&-\xi_1n_\mu\big((\partial_0x^\mu)-i(\bar\theta\Gamma^{\nu\mu}n_\nu\partial_0\theta)\big)+n_\mu\partial_0A_1^\mu-n_\mu\partial_1A_0^\mu-\phi(n^2+1).\end{aligned}\tag{17}$$

Now let us going to demonstrate the dynamics of physical variables in the action. The momenta variables according to (13) and (14) are

$$\begin{aligned}\mathcal{P}_\mu=\frac{\partial\mathcal{L}}{\partial(\partial^0x^\mu)}=&-\frac{g^{00}}{\sqrt{-g}}\big\{(\partial_0x_\mu)-i(\bar\theta\Gamma^\nu_\mu n_\nu\partial_0\theta)\big\}\\&-\frac{g^{01}}{\sqrt{-g}}\big\{(\partial_1x_\mu)-i(\bar\theta\Gamma^\nu_\mu n_\nu\partial_1\theta)\big\}+i\bar\theta\Gamma_\mu\partial_1\theta-\xi_1n_\mu,\end{aligned}\tag{18}$$

$$\pi_\theta=\frac{\partial_r\mathcal{L}}{\partial(\partial_0\theta)}=-\big\{\mathcal{P}_\mu(i\bar\theta\Gamma^{\nu\mu}n_\nu)+\bar\theta\Gamma_\mu(\partial_1x^\mu)\big\}=-\mathcal{H}_\theta,\tag{19}$$

$$\bar\pi_{\bar\theta}=\frac{\partial_rL}{\partial(\partial^0\bar\theta)}=0=-H_{\bar\theta},\tag{20}$$

$$\pi_{g_{ab}}=\frac{\partial\mathcal{L}}{\partial(\partial_0g^{ab})}=0=-\mathcal{H}_{g_{ab}},\tag{21}$$

$${\mathcal{P}_0}^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_0A_0^\mu)}=0=-H_{A_0^\mu},\tag{22}$$

$${\mathcal{P}_1}^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_0A_1^\mu)}=n_\mu=-H_{A_1^\mu},\tag{23}$$

$$\mathcal{P}_{\xi_0}=\frac{\partial\mathcal{L}}{\partial(\partial_0\xi_0)}=0=-H_{\xi_0},\tag{24}$$

$$\mathcal{P}_{\xi_1}=\frac{\partial\mathcal{L}}{\partial(\partial_0\xi_1)}=0=-H_{\xi_1},\tag{25}$$

$$\mathcal{P}_n^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 n_\mu)} = 0 = -H_{n^\mu}, \tag{26}$$

and

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = 0 = -H_\phi. \tag{27}$$

We can solve (18) for $\dot{x}_\mu$ in terms of $\mathcal{P}_\mu$ and other coordinates

$$\begin{aligned}\dot{x}_\mu \equiv \partial_0 x_\mu = -\frac{\sqrt{-g}}{g^{00}}\bigg\{\mathcal{P}_\mu + \frac{g^{01}}{\sqrt{-g}}\big\{(\partial_1 x_\mu) - i(\bar{\theta}\Gamma^\nu_\mu n_\nu \partial_1\theta)\big\} \\ - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\bigg\} + i(\bar{\theta}\Gamma^\nu_\mu n_\nu \partial_0\theta).\end{aligned} \tag{28}$$

The canonical Hamiltonian density is obtained as

$$\begin{aligned}\mathcal{H}_0 &= \mathcal{P}_\mu(\partial^0 x^\mu) + \pi_\theta(\partial_0\theta) + \bar{\pi}_{\bar{\theta}}(\partial_0\bar{\theta}) + \pi_{g_{ab}}(\partial_0 g^{ab}) \\ &\quad + \mathcal{P}_0{}^\mu(\partial_0 A_0^\mu) + \mathcal{P}_1{}^\mu(\partial_0 A_1^\mu) + \mathcal{P}_{\xi_0}(\partial_0\xi_0) \\ &\quad + \mathcal{P}_{\xi_1}(\partial_0\xi_1) + \mathcal{P}^\mu{}_n(\partial_0 n^\mu) + \pi_\phi(\partial_0\phi) - \mathcal{L} \\ &= \left(-\frac{\sqrt{-g}}{2g^{00}}\right)\bigg\{\mathcal{P}_\mu + \left(\frac{g^{01}}{\sqrt{-g}}\right)\big((\partial_1 x_\mu) - i(\bar{\theta}\Gamma^\nu_\mu n_\nu\partial_1\theta)\big) - i\bar{\theta}\Gamma_\mu\partial_1\theta \\ &\quad + \xi_1 n_\mu\bigg\}^2 + \left(-\frac{g^{11}}{2\sqrt{-g}}\right)\big((\partial_1 x^\mu)^2 - 2i(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta)(\partial_1 x_\mu)\big) \\ &\quad - \xi_0 n_\mu\big(\partial_1 x^\mu - i(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta)\big) + n_\mu\partial_1 A_0^\mu + \phi(n^2+1).\end{aligned} \tag{29}$$

The two dimensional metric is

$$g^{ab} = \begin{pmatrix} g^{00} & g^{01} \\ g^{10} & g^{11}\end{pmatrix}, \tag{30}$$

with $|g^{ab}| = g = g^{00}g^{11} - (g^{01})^2$, and considering $N = \frac{\sqrt{-g}}{g^{00}}$ and $N_1 = \frac{g^{01}}{g^{00}}$, then the canonical Hamiltonian density becomes

$$\begin{aligned}\mathcal{H}_0 = -\frac{N}{2}\big\{(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)^2 - \Pi_{1\mu}\Pi_1^\mu\big\} \\ - N_1(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)\Pi_1^\mu - \xi_0 n_\mu(\partial_1 x^\mu) \\ - i\xi_0 n_\mu(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta) + n_\mu\partial_1 A_0^\mu + \phi(n^2+1).\end{aligned} \tag{31}$$

The canonical Hamiltonian is given by

$$\begin{aligned}H_0 &= \int d\sigma\mathcal{H}_0 \\ &= \int d\sigma\bigg\{-\frac{N}{2}\big\{(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)^2 - \Pi_{1\mu}\Pi_1^\mu\big\} \\ &\quad - N_1(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)\Pi_1^\mu - \xi_0 n_\mu(\partial_1 x^\mu)\end{aligned}$$

$$- i\xi_0 n_\mu(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta) + n_\mu\partial_1 A_0^\mu + \phi(n^2+1)\Big\}. \tag{32}$$

The complete system of constraints can be presented in the form of the set of HJPDE's

$$\begin{aligned}
\mathcal{H}'_0 &= \mathcal{P}_0 - \frac{N}{2}\big\{(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)^2 - \Pi_{1\mu}\Pi_1^\mu\big\} \\
&\quad - N_1(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)\Pi_1^\mu - \xi_0 n_\mu(\partial_1 x^\mu) \\
&\quad - i\xi_0 n_\mu(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta) + n_\mu\partial_1 A_0^\mu + \phi(n^2+1), && (33)\\
\mathcal{H}'_\theta &= \pi_\theta + \big(\mathcal{P}_\mu(i\bar{\theta}\Gamma^{\nu\mu}n_\nu) + \bar{\theta}\Gamma_\mu(\partial_1 x^\mu)\big) = 0, && (34)\\
\mathcal{H}'_{\bar{\theta}} &= \bar{\pi}_{\bar{\theta}} = 0, && (35)\\
\mathcal{H}'_{g^{ab}} &= \pi_{g^{ab}} = 0, && (36)\\
\mathcal{H}'_{A_0^\mu} &= \mathcal{P}_0{}^\mu = 0, && (37)\\
\mathcal{H}'_{A_1^\mu} &= \mathcal{P}_1{}^\mu - n_\mu = 0, && (38)\\
\mathcal{H}'_{\xi_0} &= \mathcal{P}_{\xi_0} = 0, && (39)\\
\mathcal{H}'_{\xi_1} &= \mathcal{P}_{\xi_1} = 0, && (40)\\
\mathcal{H}'_{n^\mu} &= \mathcal{P}_n^\mu = 0, && (41)
\end{aligned}$$

and

$$\mathcal{H}'_\phi = \pi_\phi = 0. \tag{42}$$

Now one can observe that the dynamics of the variables is governed by the equations of motion of the form (10) and (11) such as

$$\begin{aligned}
dx_\mu &= \big\{-N(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu) - N_1\Pi_{1\mu}\big\}d\tau + i(\bar{\theta}\Gamma_\mu^\nu n_\nu)d\theta, && (43)\\
d\mathcal{P}_\mu &= 0, && (44)\\
d\pi_\theta &= 0, && (45)\\
d\bar{\pi}_{\bar{\theta}} &= -\big\{N(\mathcal{P}_\mu + \xi_1 n_\mu)(i\Gamma^\mu\partial_1\theta) + N(\partial_1 x_\mu)(i\Gamma^{\mu\nu}n_\nu\partial_1\theta) \\
&\quad - N_1(\mathcal{P}_\mu + \xi_1 n_\mu)(i\Gamma^{\mu\nu}n_\nu\partial_1\theta) + N_1(\partial_1 x_\mu)(i\Gamma^\mu\partial_1\theta) \\
&\quad - \xi_0 n_\mu(i\Gamma^{\mu\nu}n_\nu\partial_1\theta)\big\}d\tau - \big\{\mathcal{P}_\mu(i\Gamma^{\mu\nu}n_\nu) + (\partial_1 x_\mu)(i\Gamma^\mu)\big\}d\theta, && (46)\\
d\pi_{g^{ab}} &= \frac{1}{2}\bigg\{\big[(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)^2 + \Pi_{1\mu}\Pi_1^\mu\big]\left(\frac{\partial N}{\partial g^{ab}}\right) \\
&\quad + (\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu)\Pi_1^\mu\left(\frac{\partial N_1}{\partial g^{ab}}\right)\bigg\}d\tau, && (47)\\
d\mathcal{P}_0^\mu &= 0, && (48)\\
d\mathcal{P}_1^\mu &= 0, && (49)\\
d\mathcal{P}_{\xi_0} &= \big\{n_\mu(\partial_1 x^\mu + i\bar{\theta}\Gamma^{\mu\nu}n_\nu\partial_1\theta)\big\}d\tau, && (50)\\
d\mathcal{P}_{\xi_1} &= \big\{N(\mathcal{P}^\mu - i\bar{\theta}\Gamma^\mu\partial_1\theta + \xi_1 n^\mu)n_\mu + N_1\Pi_{1\mu}n^\mu\big\}d\tau, && (51)
\end{aligned}$$

$$
\begin{aligned}
d\mathcal{P}^{\mu}_{n} = \big\{ & N(\mathcal{P}^{\mu} - i\bar{\theta}\Gamma^{\mu}\partial_1\theta + \xi_1 n^{\mu})\xi_1 - N\partial_1 x_{\nu}(i\bar{\theta}\Gamma^{\nu\mu}\partial_1\theta) \\
& - N_1\xi_1 n_{\nu}(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta) + N_1\xi_1(\partial_1 x^{\mu} - i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) \\
& + \xi_0(\partial_1 x^{\mu} + i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) + \xi_0 n_{\nu}(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta) \\
& - \partial_1 A^{\mu}_0 - 2n^{\mu}\phi\big\} d\tau + dA^{\mu}_1,
\end{aligned} \tag{52}
$$

and

$$
d\pi_{\phi} = -(n^2 + 1)d\tau. \tag{53}
$$

To check whether the set of (43)–(53) is integrable or not, let us consider the total variations of the set of (HJPDE)'s. The variations of relations (33)–(42) is listed as follows:

$$
\begin{aligned}
d\mathcal{H}'_0 = \big\{ & N(\partial_1 x^{\mu} - i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) - N_1(\mathcal{P}^{\mu} - i\bar{\theta}\Gamma^{\mu}\partial_1\theta + \xi_1 n^{\mu}) \\
& - \xi_0 n_{\mu}\big\} d(\partial_1 x_{\mu}),
\end{aligned} \tag{54}
$$

$$
d\mathcal{H}'_{\theta} = (i\bar{\theta}\Gamma^{\mu})d(\partial_1 x_{\mu}), \tag{55}
$$

$$
\begin{aligned}
d\mathcal{H}'_{\bar{\theta}} = -\big\{ & N\big[\mathcal{P}_{\mu}(i\Gamma^{\mu}\partial_1\theta) + \xi_1 n_{\mu}(i\Gamma^{\mu}\partial_1\theta) + (\partial_1 x_{\mu})(i\Gamma^{\mu\nu}n_{\nu}\partial_1\theta)\big] \\
& - N_1\big[\mathcal{P}_{\mu}(i\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) + \xi_1 n_{\mu}(i\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) - (\partial_1 x_{\mu})(i\Gamma^{\mu}\partial_1\theta)\big] \\
& - \xi_0 n_{\mu}(i\Gamma^{\mu\nu}n_{\nu}\partial_1\theta)\big\} d\tau \equiv \mathcal{H}''_{\bar{\theta}} d\tau,
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
d\mathcal{H}'_{g^{ab}} = & \frac{1}{2}\Bigg\{\big[(\mathcal{P}_{\mu} - i\bar{\theta}\Gamma_{\mu}\partial_1\theta + \xi_1 n_{\mu})^2 + \Pi_{1\mu}\Pi^{\mu}_1\big]\left(\frac{\partial N}{\partial g^{ab}}\right) \\
& + \big[(\mathcal{P}_{\mu} - i\bar{\theta}\Gamma_{\mu}\partial_1\theta + \xi_1 n_{\mu})\Pi^{\mu}_1\big]\left(\frac{\partial N_1}{\partial g^{ab}}\right)\Bigg\} d\tau \\
\equiv & \mathcal{H}''_{g^{ab}} d\tau,
\end{aligned} \tag{57}
$$

$$
d\mathcal{H}'_{\xi_0} = \big\{n_{\mu}(\partial_1 x^{\mu} + i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}\partial_1\theta)\big\} d\tau \equiv \mathcal{H}''_{\xi_0} d\tau \tag{58}
$$

$$
d\mathcal{H}'_{\xi_1} = \big\{N(\mathcal{P}^{\mu} - i\bar{\theta}\Gamma^{\mu}\partial_1\theta + \xi_1 n^{\mu})n_{\mu} + N_1\Pi_{1\mu}n^{\mu}\big\} d\tau \equiv \mathcal{H}''_{\xi_1} d\tau, \tag{59}
$$

$$
d\mathcal{H}'_{A^{\mu}_0} = 0, \tag{60}
$$

$$
d\mathcal{H}'_{A^{\mu}_1} = 0, \tag{61}
$$

$$
\begin{aligned}
d\mathcal{H}'_{n^{\mu}} = \big\{ & N(\mathcal{P}^{\mu} - i\bar{\theta}\Gamma^{\mu}\partial_1\theta + \xi_1 n^{\mu})\xi_1 - N\partial_1 x_{\nu}(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta) \\
& - N_1\xi_1 n_{\mu}(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta) + N_1\xi_1(\partial_1 x^{\mu} - i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) \\
& + \xi_0(\partial_1 x^{\mu} + i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) + \xi_0 n_{\nu}(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta) - \partial_1 A^{\mu}_0 \\
& - 2n^{\mu}\phi\big\} d\tau \equiv \mathcal{H}''_{n^{\mu}} d\tau,
\end{aligned} \tag{62}
$$

$$
d\mathcal{H}'_{\phi} = -(n^2 + 1)d\tau \equiv \mathcal{H}''_{\phi} d\tau. \tag{63}
$$

The variations of the constraints $\mathcal{H}''_{\bar{\theta}}$, $\mathcal{H}''_{g^{ab}}$, $\mathcal{H}''_{\xi_0}$, $\mathcal{H}''_{\xi_1}$, $\mathcal{H}''_{n^{\mu}}$ and $\mathcal{H}''_{\phi}$ are

$$
d\mathcal{H}''_{\bar{\theta}} = -\big\{N(i\Gamma^{\mu\nu}n_{\nu}\partial_1\theta) - N_1(i\Gamma^{\mu}\partial_1\theta)\big\} d(\partial_1 x_{\mu}), \tag{64}
$$

$$
d\mathcal{H}''_{g^{ab}} = \Bigg\{(\partial_1 x^{\mu} + i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}\partial_1\theta)\left(\frac{\partial N}{\partial g^{ab}}\right)
$$

$$+(\mathcal{P}^{\mu}-i\bar{\theta}\Gamma^{\mu}\partial_1\theta+\xi_1 n^{\mu})\left(\frac{\partial N_1}{\partial g^{ab}}\right)\bigg\}d(\partial_1 x_{\mu}), \tag{65}$$

$$d\mathcal{H}''_{\xi_0}=n_{\mu}d(\partial_1 x^{\mu}), \tag{66}$$

$$d\mathcal{H}''_{\xi_1}=N_1 n_{\mu}d(\partial_1 x^{\mu}), \tag{67}$$

$$d\mathcal{H}''_{n^{\mu}}=\left\{N(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta)-N_1\xi_1-\xi_0\right\}d(\partial_1 x_{\mu}), \tag{68}$$

and

$$d\mathcal{H}''_{\phi}=0. \tag{69}$$

However, the variations of constraints $\mathcal{H}'_0$, $\mathcal{H}'_{\theta}$, $\mathcal{H}''_{\bar{\theta}}$, $\mathcal{H}''_{g^{ab}}$, $\mathcal{H}''_{\xi_0}$, $\mathcal{H}''_{\xi_1}$ and $\mathcal{H}''_{n^{\mu}}$ do not vanish identically, a new set of constraints arises. According to the theory, this new set of constraints is added to the equations of motion of the system.

Using the open string boundary conditions [35], $x'^{\mu}=\partial_1 x_{\mu}=0$, for $\sigma=0$ and $\sigma=\pi$ the variations of the new constraints (64)–(69) are identically zero, and the system is integrable.

## 4 Hamilton-Jacobi Formulation of $D=11$ Superstring

In this section we study the superstring problem in a more simple framework of mechanical model. Our starting point is the Lagrangian action [34],

$$S=\int d\tau\left\{\frac{1}{2e}\Pi_{\mu}\Pi^{\mu}+n_{\mu}\dot{z}^{\mu}-\frac{1}{2}\phi(n^2+1)\right\}, \tag{70}$$

with

$$\Pi^{\mu}\equiv\dot{x}^{\mu}-i(\bar{\theta}\Gamma^{\nu\mu}\dot{\theta})n_{\nu}-\xi n^{\mu}, \tag{71}$$

where $x^{\mu}$, $n^{\mu}$, $e$, $\phi$ and $\xi$ are Grassmann even variables and $\theta^{\alpha}$ are Grassmann odd variables, dependent on the evolution parameter $\tau$. The singularity of the Lagrangian

$$L=\frac{1}{2e}\Pi_{\mu}\Pi^{\mu}+n_{\mu}\dot{z}^{\mu}-\phi(n^2+1), \tag{72}$$

follows from the fact that the rank of the Hess matrix $A_{ij}$ is one.

The canonical momenta according to (2) and (3) read as

$$P^{\mu}=\frac{\partial L}{\partial\dot{x}^{\mu}}=\frac{1}{e}\left\{\dot{x}_{\mu}-i(\bar{\theta}\Gamma^{\mu\nu}\dot{\theta})n_{\nu}-\xi n^{\mu}\right\}, \tag{73}$$

$$\pi_{\theta}=\frac{\partial_r L}{\partial\dot{\theta}}=-i(\bar{\theta}\Gamma^{\mu\nu})n_{\nu}P_{\mu}=-H_{\theta}, \tag{74}$$

$$\pi_{\bar{\theta}}=\frac{\partial_r L}{\partial\dot{\bar{\theta}}}=0=-H_{\bar{\theta}}, \tag{75}$$

$$P_n^{\mu}=\frac{\partial L}{\partial\dot{n}^{\mu}}=0=-H_n^{\mu}, \tag{76}$$

$$P_{\xi}=\frac{\partial L}{\partial\dot{\xi}}=0=-H_{\xi}, \tag{77}$$

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e, \tag{78}$$

$$P_z^{\mu} = \frac{\partial L}{\partial \dot{z}_{\mu}} = n^{\mu} = -H_z^{\mu}, \tag{79}$$

and

$$\pi_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = 0 = -H_{\phi}. \tag{80}$$

Since the rank of the Hess matrix is one, we can solve (73) for $\dot{x}^{\mu}$ in terms of $P^{\mu}$ and other coordinates

$$\dot{x}^{\mu} = eP^{\mu} + i\big(\bar{\theta}\Gamma^{\mu\nu}\dot{\theta}\big)n_{\nu} + \xi n^{\mu}. \tag{81}$$

A straightforward calculation shows that the canonical Hamiltonian $H_0$ is obtained as

$$H_0 = \frac{1}{2}eP^2 + \phi(n^2 + 1) + P^{\mu}\xi n_{\mu}. \tag{82}$$

Following the Hamilton-Jacobi formalism we obtain the set of (HJPDE)'s,

$$H_0' = P_0 + \frac{1}{2}eP^2 + \phi(n^2 + 1) + P^{\mu}\xi n_{\mu}, \tag{83}$$

$$H_{\theta}' = \pi_{\theta} + i\bar{\theta}\Gamma^{\mu\nu}n_{\nu}P_{\mu}, \tag{84}$$

$$H_{\bar{\theta}}' = \pi_{\bar{\theta}}, \tag{85}$$

$$H_n'^{\mu} = P_n^{\mu}, \tag{86}$$

$$H_{\xi}' = P_{\xi}, \tag{87}$$

$$H_e' = P_e, \tag{88}$$

$$H_z'^{\mu} = P_z^{\mu} - n^{\mu}, \tag{89}$$

and

$$H_{\phi}' = \pi_{\phi}. \tag{90}$$

Therefore, the total differential equations for the characteristic (10) and (11) read as

$$dx^{\mu} = (eP^{\mu} + \xi n^{\mu})d\tau + i(\bar{\theta}\Gamma^{\mu\nu}n_{\nu})d\theta, \tag{91}$$

$$dP_0 = 0, \tag{92}$$

$$dP^{\mu} = 0, \tag{93}$$

$$d\pi_{\theta} = 0, \tag{94}$$

$$d\pi_{\bar{\theta}} = (iP_{\mu}\Gamma^{\mu\nu}n_{\nu})d\theta, \tag{95}$$

$$dP_n^{\mu} = -(2\phi n^{\mu} + \xi P^{\mu})d\tau - (iP_{\mu}\Gamma^{\mu\nu})d\theta + dz^{\mu}, \tag{96}$$

$$dP_{\xi} = -(P_{\mu}n^{\mu})d\tau, \tag{97}$$

$$dP_e = -\left(\frac{1}{2}P^2\right)d\tau, \tag{98}$$

$$dP_z^{\mu} = 0, \tag{99}$$

and

$$dP_\phi = -(n^2+1)d\tau. \tag{100}$$

To check whether the set of (91)–(100) are integrable or not, let us consider the total variations of the set of (HJPDE)'s. The variations of

$$dH_0' = 0, \tag{101}$$

$$dH_\theta' = 0, \tag{102}$$

$$dH_{\bar{\theta}}' = 0, \tag{103}$$

and

$$dH_z'^{\mu} = 0, \tag{104}$$

are identically zero, whereas the variations of

$$dH_n'^{\mu} = -(2\phi n_\mu + \xi P_\mu)d\tau - (iP_\mu\Gamma^{\mu\nu})d\theta + dz^\mu \equiv H_n''^{\mu} d\tau, \tag{105}$$

$$dH_\xi' = -(P_\mu n^\mu)d\tau \equiv H_\xi'' d\tau, \tag{106}$$

$$dH_e' = -\left(\frac{1}{2}P^2\right)d\tau \equiv H_e'' d\tau, \tag{107}$$

and

$$dH_\phi' = -(n^2+1)d\tau \equiv H_\phi'' d\tau \tag{108}$$

are not. Therefore we obtain the following set of additional constraints:

$$H_n''^{\mu} = -(2\phi n_\mu + \xi P_\mu), \tag{109}$$

$$H_\xi'' = -(P_\mu n^\mu), \tag{110}$$

$$H_7'' = -\left(\frac{1}{2}P^2\right), \tag{111}$$

and

$$H_9'' = -(n^2+1). \tag{112}$$

One notices that the total differentials of $H_n''^{\mu}$, $H_\xi''$, $H_e''$ and $H_\phi''$ vanish identically, i.e.

$$dH_n''^{\mu} = 0, \tag{113}$$

$$dH_\xi'' = 0, \tag{114}$$

$$dH_e'' = 0, \tag{115}$$

and

$$dH_\phi'' = 0. \tag{116}$$

Thus the equations of motion (91)–(100) and the new constraints (109)–(112) represent an integrable system. Since the equations of motion are integrable, the action can be written as

$$S = \int d\tau \left\{ \frac{1}{2}eP^2 + n_\mu \dot{z}^\mu - \phi(n^2+1) \right\}. \tag{117}$$

## 5 Conclusion

We have performed the detailed analysis of the Green-Schwarz super-string model, by applying Hamilton-Jacobi formalism. A very interesting and intriguing observation is that the Hamilton-Jacobi formalism efficient to solve the supersymmetric constraints system; which contains the fermionic degrees of freedom in the Hamiltonian formulation yields fermionic constraints, the half of these are first-class and the remaining half are second-class constraints. The constraints are treated with no need to introduced fixation gauge or to replace the second-class constraints with the first-class ones [36]. The system is completely integrable if and only if we use the boundary condition of the string as a constraint on such system. Finally, we present $D = 11$ action for mechanical superstring system, the constraints of a dynamical system are expressed as the set of (HJPDE)'s, then the equations of motion are obtained. The integrability conditions are examined until a complete system is obtained.

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